

Hardy's proof of nonlocality in the presence of noise

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We extend the validity of Hardy's nonlocality without inequalities proof to cover the case of special one-parameter classes of nonpure statistical operators. These mixed states are obtained by mixing the Hardy states with a completely chaotic noise or with a colored noise and they represent a realistic description of imperfect preparation processes of (pure) Hardy states in nonlocality experiments. Within such a framework we are able to exhibit a precise range of values of the parameter measuring the noise affecting the non-optimal preparation of an arbitrary Hardy state, for which it is still possible to put into evidence genuine nonlocal effects. Equivalently, our work exhibits particular classes of bipartite mixed states whose constituents do not admit any local and deterministic hidden variable model reproducing the quantum mechanical predictions.

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I. INTRODUCTION

Nonlocality without inequalities arguments, like the celebrated Greenberger-Horne-Zeilinger (GHZ) [1] and Hardy's [2] ones, may provide evidence of the genuine nonlocal features of certain quantum states without resorting to the consideration of Bell-like inequalities [3, 4]. The experimental verification of the implications of such arguments does not require one to collect data of several correlation functions but it simply demands one to test the occurrence of certain joint measurement outcomes. More precisely, in the case of the Greenberger-Horne-Zeilinger argument one has to verify the existence of perfect (anti)correlations between the outcomes of dichotomic measurements performed on to each of a group of three spacelike separated particles [5, 6], while in the case of Hardy's proof one has to put into evidence the occurrence of a single joint event [7, 8]. Such experiments usually involve groups of polarization-entangled photons and the most reliable source of (multipartite) entanglement is represented by parametric down conversion processes [9]. However, in practice, the pure states exhibiting nonlocal effects one wishes to prepare in a laboratory in this way (the tripartite GHZ state and the Hardy states, which are bipartite state vectors whose Schmidt decomposition involves at least two different weights) are unavoidably subjected to different kinds of noise. The aim of this paper is to show how it is possible to generalize the original Hardy's proof [2] by following techniques which are similar to those we used in Refs. [10, 11] and to apply it to specific one-parameter classes of statistical operators, representing mixtures of Hardy states with a completely chaotic noise (also known as a white noise) or with a colored noise. While the first kind of noise the one which is usually considered in the literature for nonlocality experiments, the second kind has been recently pointed out to be the best (and more realistic) choice for describing what really happens in type II spontaneous parametric down conversion processes [12].

Therefore, in this paper, we will determine the range of values of the parameter which measures the amount of noise affecting the nonoptimal preparation of a definite (pure) Hardy state, for which it is still possible that the resulting mixed state exhibits genuine and testable nonlocal effects. Equivalently, our work exhibits particular classes of bipartite mixed states whose constituents do not admit any local and deterministic hidden variable model which may consistently be in agreement with the quantum mechanical predictions, and which are, as a consequence, nonseparable (that is, they cannot be expressed as a convex sum of product states). As already remarked on, the techniques we use are similar to those presented in Ref. [11] but some results are improved because in this paper we exploit the explicit form of the noise corrupting a Hardy state, while in Ref. [11] the form of the mixed states we were considering was left unspecified for the sake of generality.

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II. HIDDEN VARIABLE MODELS

In ordinary quantum mechanics, given the state vector associated to a physical system, the outcomes of (single and joint) measurement processes of arbitrary observables may be only statistically predicted and, apart from particular situations, such outcomes are not certain. However, there still exists the (logical) possibility that the quantum states do not represent the maximal possible knowledge one can have about a quantum system. In such a situation, supplementary variables might very well predetermine the outcomes of any quantum measurement and the stochastic aspects of quantum mechanics would arise since what one can know is only the probabilistic distribution of these additional variables—for this reason they are referred to as hidden variables.

Accordingly, the hypothetical theory (called stochastic hidden variable model) which completes quantum mechanics consists of (i) a set Λ , whose elements λ are the hidden variables; (ii) a normalized probability distribution ρ defined on Λ ; (iii) a set of probability distributions $P_\lambda(A_i = a, B_j = b, \dots, Z_k = z)$ for the outcomes of single and joint measurements of any conceivable set of observables $\{A_i, B_j, \dots, Z_k\}$, where each index of the set $\{i, j, \dots, k\}$ refers to a single particle or to a group of all the particles, such that

$$P_\sigma(A_i = a, B_j = b, \dots, Z_k = z) = \int_\Lambda d\lambda \rho(\lambda) P_\lambda(A_i = a, B_j = b, \dots, Z_k = z). \quad (1)$$

The quantity at the left hand side of Eq. (1) is the quantum mechanical (joint) probability distribution for the set of outcomes $\{a, b, \dots, z\}$ of the considered measurements when the system is in the state σ . A deterministic hidden variable model, which is also known as a realistic model, is a particular instance of a hidden variable model where all probabilities P_λ take the values 0 or 1 only. In this case, the measurement outcomes of arbitrary observables are predetermined.

A hidden variable model is called local [13] if the following factorizability condition holds for any conceivable joint probability distribution $P_\lambda(A_i = a, B_j = b, \dots, Z_k = z)$ and for any value of the hidden variable $\lambda \in \Lambda$

$$P_\lambda(A_i = a, B_j = b, \dots, Z_k = z) = P_\lambda(A_i = a) P_\lambda(B_j = b) \dots P_\lambda(Z_k = z) \quad (2)$$

in all cases in which the measurement processes for the observables A_i, B_j, \dots, Z_k occur at spacelike separated locations. The locality condition imposes that no causal influence can exist between spacelike separated events. Since deterministic and stochastic hidden variable models are totally equivalent when the locality condition is imposed [14], and since in what follows we will focus only on realistic models, all the probabilities P_λ will consequently be assumed to take the values 0 and 1 only.

III. HARDY'S PROOF IN THE PRESENCE OF NOISE

A Hardy state $|\psi\rangle$ belonging to $\mathbb{C}^2 \otimes \mathbb{C}^2$ is any entangled state whose Schmidt decomposition with respect to an orthonormal basis $\{|0\rangle, |1\rangle\}$, involves two (strictly positive) different weights $p_1 \neq p_2$ so that

$$|\psi\rangle = p_1|0\rangle \otimes |0\rangle + p_2|1\rangle \otimes |1\rangle, \quad p_1^2 + p_2^2 = 1. \quad (3)$$

Thus, the Hardy states of Eq. (3) are simply all entangled but not maximally entangled two-qubit states. By following Ref. [10] we define two orthonormal bases, depending on the Schmidt coefficients p_1 and p_2 of Eq. (3), $\{|x_+\rangle, |x_-\rangle\}$ and $\{|y_+\rangle, |y_-\rangle\}$ spanning \mathbb{C}^2 , as follows:

$$\begin{bmatrix} |x_+\rangle \\ |x_-\rangle \end{bmatrix} = \frac{1}{\sqrt{p_1 + p_2}} \begin{bmatrix} \sqrt{p_2} & -i\sqrt{p_1} \\ -i\sqrt{p_1} & \sqrt{p_2} \end{bmatrix} \begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} |y_+\rangle \\ |y_-\rangle \end{bmatrix} = \frac{1}{\sqrt{(p_1^2 + p_2^2 - p_1 p_2)(p_1 + p_2)}} \begin{bmatrix} -ip_2\sqrt{p_2} & p_1\sqrt{p_1} \\ p_1\sqrt{p_1} & -ip_2\sqrt{p_2} \end{bmatrix} \begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix}. \quad (5)$$

Then, we denote as X_i and Y_i (where $i = 1, 2$ is the particle index) the observables whose eigenstates associated to the eigenvalues $+1$ and -1 are the vectors $\{|x_+\rangle, |x_-\rangle\}$ and $\{|y_+\rangle, |y_-\rangle\}$ of Eqs. (4) and (5), respectively.

State vectors $|\psi\rangle$ of the kind of Eq. (3) are known to exhibit nonlocal features [2] because certain joint probability distributions involving the observables X_i and Y_i turn out to be locally inexplicable. The aim of this paper is to prove that similar nonlocal effects arise also when considering particular classes of mixed states generated from the Hardy

states of Eq. (3). To start with, let us consider the mixed statistical operator obtained by taking the convex sum of a pure Hardy state of Eq. (3) with a completely chaotic noise (also called white noise) as

$$\sigma = p|\psi\rangle\langle\psi| + \frac{1-p}{4}I_2 \otimes I_2, \quad (6)$$

where I_2 is the identity operator in \mathbb{C}^2 and $p \in [0, 1]$ is the parameter which measures the amount of noise affecting the purity of the Hardy state $|\psi\rangle$. Such a statistical operator represents the state which usually results from an imperfect preparation of the (pure) Hardy state $|\psi\rangle$ in the practical realizations of nonlocality experiments. In fact, it is customary to mimic the noise which affects a nonperfectly controllable preparation of pure entangled states, like those of polarization-entangled photons produced by parametric down-conversion processes, as the output state of Eq. (6) [15].

In order to exhibit a proof of nonlocality without inequalities for the state σ by following the Hardy's argument, we have to consider appropriate quantum mechanical probability distributions and to show that they cannot be reproduced by local realistic hidden variable models. To this end let us briefly review the line of reasoning of Ref. [11] and consider the observables X_i and Y_i , defined above, and we take into account the following joint-probability distributions when the state of the system is that of Eq. (6):

$$P_\sigma(X_1 = +1, X_2 = +1) = \frac{1-p}{4} \equiv \varepsilon, \quad (7)$$

$$P_\sigma(Y_1 = +1, X_2 = -1) = \frac{1-p}{4} \equiv \varepsilon, \quad (8)$$

$$P_\sigma(X_1 = -1, Y_2 = +1) = \frac{1-p}{4} \equiv \varepsilon, \quad (9)$$

$$P_\sigma(Y_1 = +1, Y_2 = +1) = p \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(1 - p_1 p_2)^2} + \frac{1-p}{4} \equiv a + \varepsilon, \quad (10)$$

where the quantity $\frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(1 - p_1 p_2)^2}$ does not vanish (in fact, by hypothesis, p_1 and p_2 are strictly positive and different from each other). This quantity represents the crucial joint probability which allows one to establish the nonlocal features of any Hardy state in the original formulation of the argument of nonlocality [2, 10].

Now we are going to determine which are the allowed values for the parameter p if a local and realistic model exists which is able to reproduce the probabilities of Eqs. (7)-(10). To this purpose, suppose that a local and realistic description (as defined in the previous section) exists for the state σ and consider what happens, for example, with the probability distribution $P_\sigma(X_1 = +1, X_2 = +1)$. According to the definition of a local hidden variable model, we have

$$P_\sigma(X_1 = +1, X_2 = +1) = \int_\Lambda d\lambda \rho(\lambda) P_\lambda(X_1 = +1) P_\lambda(X_2 = +1). \quad (11)$$

Since we are dealing with a deterministic model, where the (single-particle) probabilities P_λ possess values 0 or 1 only, it is useful to define the following subsets A, B, C , and D of Λ as

$$A = \{ \lambda \in \Lambda \mid P_\lambda(X_1 = +1) = 1 \}, \quad (12)$$

$$B = \{ \lambda \in \Lambda \mid P_\lambda(X_2 = +1) = 1 \}, \quad (13)$$

$$C = \{ \lambda \in \Lambda \mid P_\lambda(Y_1 = +1) = 1 \}, \quad (14)$$

$$D = \{ \lambda \in \Lambda \mid P_\lambda(Y_2 = +1) = 1 \}. \quad (15)$$

If we denote as $\mu(\Sigma)$ the measure of any subset Σ of Λ with respect to the weight function $\rho(\lambda)$, i.e., $\mu(\Sigma) = \int_\Sigma d\lambda \rho(\lambda)$, the probability distributions of Eqs. (7)-(10) turn out to be equivalent to

$$\mu[A \cap B] = \varepsilon, \quad (16)$$

$$\mu[C] - \mu[B \cap C] = \varepsilon, \quad (17)$$

$$\mu[D] - \mu[A \cap D] = \varepsilon, \quad (18)$$

$$\mu[C \cap D] = a + \varepsilon. \quad (19)$$

If we follow the set-theoretic manipulations presented in Ref. [11], starting from Eqs. (16)-(19), we end up with an inequality constraining the values of ε and a , as long as a local realistic model for σ is supposed to exist, written

$$2\varepsilon - a \geq 0. \quad (20)$$

This relation, when expressed in terms of the parameters p , p_1 and p_2 , takes the following form

$$0 \leq p \leq \frac{1}{1 + 2 \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(1 - p_1 p_2)^2}}. \quad (21)$$

To summarize, we have proven the following theorem:

Theorem I. Consider the normalized entangled vector $|\psi\rangle = p_1|0\rangle|0\rangle + p_2|1\rangle|1\rangle$ belonging to $\mathbb{C}^2 \otimes \mathbb{C}^2$, with different (strictly positive) weights $p_1 \neq p_2$, and the one-parameter class of mixed statistical operators $\sigma = p|\psi\rangle\langle\psi| + \frac{1-p}{4}I_2 \otimes I_2$, where $p \in [0, 1]$. If there exists a local and deterministic hidden variable model for σ then $p \in [0, \frac{1}{1 + 2 \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(1 - p_1 p_2)^2}}]$.

Thus, we have succeeded in exhibiting a necessary condition for the existence of a local realistic model for the one-parameter class of operators σ of Eq. (6). The usefulness of this result is twofold, both from the theoretical and from the experimental point of view, as can be immediately deduced by reversing the assertion of the previous theorem.

In fact, first, for any $p \in (\frac{1}{1 + 2 \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(1 - p_1 p_2)^2}}, 1]$ the associated mixed states cannot be locally described in terms of a deterministic hidden variable model where all the measurement outcomes are predetermined. As a consequence, such states are proven to be not separable, that is, they cannot be decomposed as a convex sum of product states (indeed, if they were separable a local realistic model for them would actually exist [16]).

Second, for the same values of p , genuine nonlocal effects can be successfully revealed by experiments where joint measurements are performed by spacelike separated observers, despite the presence of a completely chaotic noise corrupting a pure Hardy state. Stated equivalently, we have been able to obtain a (not necessarily optimal) bound for the maximum amount of white noise affecting any state $|\psi\rangle$ of the type of Eq. (3), in the presence of which it is still possible to exhibit a Hardy's proof of nonlocality.

To give a numerical example let us consider a (pure) Hardy state $|\psi\rangle$ of Eq. (3) for which $p_1 p_2 = (3 - \sqrt{5})/2$. In this case the probability $P_\psi(Y_1 = +1, Y_2 = +1) = \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(1 - p_1 p_2)^2}$ attains its maximum value, approximately equal to 0.09. As a consequence, in this situation one obtains the maximal violation of the locality condition allowed by the original Hardy's proof [2]. By using the result of Eq. (21) and the above value for $p_1 p_2$, we can conclude that, as long as the parameter p belongs to the interval $p \in (0.85, 1)$ —the inferior value of this interval being a two-digit approximation of the exact value one can deduce from Eq. (21)— one can still put experimentally into evidence nonlocal effects despite the presence of a white noise affecting the preparation of the pure Hardy state $|\psi\rangle$.

Let us pass now to analyze what happens if we replace the white noise considered so far with another kind of noise which has been recently suggested [12] to be a more realistic description for a noise affecting the preparation of entangled states which are generated through parametric-down conversion processes. It is referred to as a colored noise and the one-parameter class of mixed states we will now consider is

$$\sigma = p|\psi\rangle\langle\psi| + \frac{1-p}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|) \quad (22)$$

where, once again, the parameter p measures the degree of purity of the state and it belongs to the interval $[0, 1]$. Given the states of Eq. (22), we can calculate the (modified) joint probabilities for the set of observables X_i and Y_i obtaining:

$$P_\sigma(X_1 = +1, X_2 = +1) = \frac{1-p}{2(p_1 + p_2)^2} \equiv \varepsilon_1, \quad (23)$$

$$P_\sigma(Y_1 = +1, X_2 = -1) = \frac{(1-p)p_1 p_2}{2(p_1 + p_2)^2(1 - p_1 p_2)} \equiv \varepsilon_2, \quad (24)$$

$$P_\sigma(X_1 = -1, Y_2 = +1) = \frac{(1-p)p_1 p_2}{2(p_1 + p_2)^2(1 - p_1 p_2)} \equiv \varepsilon_2, \quad (25)$$

$$P_\sigma(Y_1 = +1, Y_2 = +1) = \frac{1 - 3p_1^2 p_2^2 + p(-8p_1^4 p_2^4 + 5p_1^2 p_2^2 - 1)}{2(p_1 + p_2)^2(1 - p_1 p_2)^2} \equiv \varepsilon_3. \quad (26)$$

If we suppose once again that there exists a local and deterministic hidden variable model able to reproduce the probability distributions of Eqs. (23)-(26), and if we define the subsets A, B, C , and D as before and follow the reasonings presented in Ref. [11], we end up with an inequality constraining the values of $\varepsilon_1, \varepsilon_2$, and ε_3 ,

$$\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3 \geq 0. \quad (27)$$

As a consequence, by using the definition of ε_i as given in Eqs. (23)-(26), we obtain a relation, equivalent to that of Eq. (27), in terms of the parameters p_1, p_2 , and p ,

$$0 \leq p \leq \frac{1}{2(1-2p_1^2p_2^2)} \quad (28)$$

and the following theorem holds:

Theorem II. Consider the normalized entangled vector $|\psi\rangle = p_1|0\rangle|0\rangle + p_2|1\rangle|1\rangle$ belonging to $\mathbb{C}^2 \otimes \mathbb{C}^2$, with different (strictly positive) weights $p_1 \neq p_2$, and the one-parameter class of mixed statistical operators $\sigma = p|\psi\rangle\langle\psi| + \frac{1-p}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$ where $p \in [0, 1]$. If there exists a local and deterministic hidden variable model for σ then $p \in [0, \frac{1}{2(1-2p_1^2p_2^2)}]$.

Stated equivalently, whenever $p \in (\frac{1}{2(1-2p_1^2p_2^2)}, 1]$, no local realistic model can exist for the corresponding mixed state, describing a Hardy state corrupted by a colored noise. As a consequence, such states cannot be separable states. From the experimental point of view, Hardy states mixed with a colored noise are more useful, with respect to those corrupted by a white noise, for what concerns the possibility of highlighting nonlocal effects. In fact, since the following inequalities

$$0 < \frac{1}{2(1-2p_1^2p_2^2)} < \frac{1}{1+2\frac{p_1^2p_2^2(p_1-p_2)^2}{(1-p_1p_2)^2}} < 1 \quad (29)$$

hold for all values of $p_1 \neq p_2$, such that $p_1^2 + p_2^2 = 1$, when one deals with a colored rather than with a white noise one obtains a larger interval of values of p , for which nonlocal effects can be experimentally put into evidence.

To give a numerical example, let us consider again the (pure) Hardy state which implies the maximal violation of the locality condition in Hardy's proof: since for it $p_1p_2 = (3 - \sqrt{5})/2$, one obtains by theorem II that for any $p \in (0.70, 1]$ it is possible to put into evidence nonlocal effects and this is to be compared with the analogous result $p \in (0.85, 1]$ we have for the case of a white noise.

Before passing to analyze what happens in Hilbert spaces of greater dimensionality, let us address the issue of how powerful the Hardy's criterion is in excluding the existence of local realistic models, for the considered mixed states of Eqs. (6) and (22), with respect to other existing criteria. In fact, besides Hardy's proof, the usual way to reject local hidden variable models for a given quantum state consists in exhibiting a Bell inequality which is violated by an appropriate choice of single-particle observables. For this purpose, and in the restricted scenario of a $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space, a necessary condition has been exhibited in Ref. [17] in order that a given mixed state satisfies all conceivable CHSH inequalities [4]. Such a condition, when applied to the states σ of Eq. (6), affirms that $p \in [0, \frac{1}{\sqrt{1+4p_1^2p_2^2}}]$ if σ does not violate any Bell-like inequality. Equivalently, one can conclude that whenever

$$p \in (\frac{1}{\sqrt{1+4p_1^2p_2^2}}, 1] \quad (30)$$

there exists at least one CHSH inequality which is violated, and, consequently, that σ cannot be described by any local realistic model. Unfortunately, since the following inequality

$$\frac{1}{\sqrt{1+4p_1^2p_2^2}} < \frac{1}{1+2\frac{p_1^2p_2^2(p_1-p_2)^2}{(1-p_1p_2)^2}} < 1, \quad (31)$$

holds for any (strictly positive) $p_1 \neq p_2$, such that $p_1^2 + p_2^2 = 1$, it turns out that the criterion of Ref. [17] is more powerful than our generalized version of Hardy's proof to deny the existence of local realistic models for the considered class of mixed states of Eq. (6). A similar conclusion can be reached when considering the states of Eq. (22). In fact, in this case, the criterion of Ref. [17] tells that for any $p \in (0, 1]$ there exists a violated CHSH inequality and, as a consequence, local realistic models cannot exist for any value of $p \in (0, 1]$, while our method individuates nonlocal states only for values of p belonging to the interval $(\frac{1}{2(1-2p_1^2p_2^2)}, 1]$.

Thus from a theoretical point of view, our argument of nonlocality is weaker than the one exhibited in Ref. [17] if consideration is given to the restricted one-parameter class of mixed states of Eqs. (6) and (22). On the contrary, from a practical point of view, the result obtained when considering white noise is of interest since, to identify and to test experimentally the crucial CHSH inequality which is violated by those states σ such that Eq. (30) holds, might turn out to be much more difficult than to perform the tests which are the crucial ones from our perspective. Moreover, better results will be achieved when we will soon consider classes of mixed states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, where d_1 and d_2 can be

any positive integer greater than or equal to two, a situation in which no criterion of the type of Ref. [17] is currently known.

To this end, let us now consider an entangled state $|\phi\rangle$ whose Schmidt decomposition in terms of appropriate orthonormal sets of states $\{|k\rangle\}$ belonging to \mathbb{C}^{d_1} and $\{|j\rangle\}$ belonging to \mathbb{C}^{d_2} , respectively, involves at least two (strictly positive) different weights –which we suppose for simplicity to be p_1 and p_2 :

$$|\phi\rangle = p_1|0\rangle \otimes |0\rangle + p_2|1\rangle \otimes |1\rangle + \sum_{i \geq 3}^{\leq \min(d_1, d_2)} p_i |i-1\rangle \otimes |i-1\rangle, \quad \sum_i p_i^2 = 1. \quad (32)$$

States like these will again be referred to as Hardy states. Subsequently, we define within the two-dimensional manifold spanned by $\{|0\rangle, |1\rangle\}$ the vectors $\{|x_+\rangle, |x_-\rangle\}$ and $\{|y_+\rangle, |y_-\rangle\}$ as in Eqs. (4)-(5). The observables X_j and Y_j (where $j = 1, 2$) are then defined as before, with the further condition that they possess the degenerate eigenvalue 0 (in addition to $+1$ and -1) whose eigenmanifold is that spanned by the vectors $\{|i-1\rangle\}$ for any $i \geq 3$.

Once again, in order to generalize the results we obtained in the case of bipartite mixed states of two spin-1/2 particles, let us consider the one-parameter class of mixed statistical operators σ , obtained by taking a convex mixture of a Hardy state $|\phi\rangle$ of Eq. (32) and a completely chaotic noise as follows:

$$\sigma = p|\phi\rangle\langle\phi| + \frac{1-p}{d_1 d_2} I_{d_1} \otimes I_{d_2}, \quad (33)$$

where I_{d_1} and I_{d_2} are the identity operator in \mathbb{C}^{d_1} and \mathbb{C}^{d_2} , respectively. In order to determine a range of values of the parameter p such that one can prove that a local and deterministic hidden variable model for the corresponding mixed states cannot exist, we need to take into account appropriate correlation functions. To this end, given the state of Eq. (33), we will consider the following joint probability distributions for the measurement outcomes of the observables X_i and Y_j

$$P_\sigma(X_1 = +1, X_2 = +1) = (1-p)/d_1 d_2 \equiv \varepsilon, \quad (34)$$

$$P_\sigma(Y_1 = +1, X_2 = -1) = (1-p)/d_1 d_2 \equiv \varepsilon, \quad (35)$$

$$P_\sigma(X_1 = -1, Y_2 = +1) = (1-p)/d_1 d_2 \equiv \varepsilon, \quad (36)$$

$$P_\sigma(Y_1 = +1, X_2 = 0) = (1-p)/d_1 d_2 \equiv \varepsilon, \quad (37)$$

$$P_\sigma(X_1 = 0, Y_2 = +1) = (1-p)/d_1 d_2 \equiv \varepsilon, \quad (38)$$

$$P_\sigma(Y_1 = +1, Y_2 = +1) = p \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(p_1^2 + p_2^2 - p_1 p_2)^2} + \frac{1-p}{d_1 d_2} \equiv a + \varepsilon. \quad (39)$$

Let us now suppose that a local realistic model may account for such joint probability distributions, define the sets A, B, C , and D as in Eqs. (12)-(15) and let us see what happens with, e.g., Eq. (35):

$$P_\sigma(Y_1 = +1, X_2 = -1) = \int_{\Lambda} d\lambda \rho(\lambda) P_\lambda(Y_1 = +1) P_\lambda(X_2 = -1) \quad (40)$$

$$= \int_{\Lambda} d\lambda \rho(\lambda) P_\lambda(Y_1 = +1) [1 - P_\lambda(X_2 = 1) + P_\lambda(X_2 = 0)] \quad (41)$$

$$= \mu[C] - \mu[B \cap C] - \varepsilon. \quad (42)$$

The second equality follows since $P_\lambda(X_2 = -1) + P_\lambda(X_2 = 0) + P_\lambda(X_2 = +1) = 1$ is a relation which holds for any $\lambda \in \Lambda$, while the third equality descends from Eq. (37). Finally, since $P_\sigma(Y_1 = +1, X_2 = -1)$ equals ε due to Eq. (35), we obtain the desired relation between the indicated subsets C and $B \cap C$ of Λ , that is

$$\mu[C] - \mu[B \cap C] = 2\varepsilon. \quad (43)$$

Similar arguments can be used with the other probability distributions of Eqs. (34)-(39) so as to obtain constraints which the measures of appropriate subsets of Λ have to satisfy if a local realistic model for σ exists, that is

$$\mu[A \cap B] = \varepsilon, \quad (44)$$

$$\mu[C] - \mu[B \cap C] = 2\varepsilon, \quad (45)$$

$$\mu[D] - \mu[A \cap D] = 2\varepsilon, \quad (46)$$

$$\mu[C \cap D] = a + \varepsilon. \quad (47)$$

These equations are similar to Eqs. (16)-(19), apart from a multiplicative factor which appears in Eqs. (45) and (46) and which is related to the fact the observables X_i and Y_j we are considering, possess now three different eigenvalues, rather than two as before. Due to this similarity, we can follow the set-theoretic manipulations we used in Ref. [11] and conclude that if a local realistic model exists for the mixed states of Eq. (33), then the following inequality, involving the parameters ε and a , has to hold

$$4\varepsilon - a \geq 0. \quad (48)$$

Taking into account the definition of ε and a given in Eqs. (34) and (39) with respect to d_1, d_2, p_1, p_2 , and p , the following theorem follows:

Theorem III. Consider the normalized entangled vector $|\phi\rangle = \sum_{i \geq 1} p_i |i-1\rangle \otimes |i-1\rangle$ belonging to $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, with different (strictly positive) weights $p_1 \neq p_2$, and the one-parameter class of mixed statistical operators $\sigma = p|\phi\rangle\langle\phi| + \frac{1-p}{d_1 d_2} I_{d_1} \otimes I_{d_2}$, where $p \in [0, 1]$. If there exists a local and deterministic hidden variable model for σ then $p \in [0, \frac{1}{1 + \frac{d_1 d_2 p_1^2 p_2^2 (p_1 - p_2)^2}{4(p_1^2 + p_2^2 - p_1 p_2)^2}}]$.

Once again, this theorem allows us to conclude that the one-parameter class of statistical operators σ of Eq. (33) with values of p such that

$$p \in (\frac{1}{1 + \frac{d_1 d_2 p_1^2 p_2^2 (p_1 - p_2)^2}{4(p_1^2 + p_2^2 - p_1 p_2)^2}}, 1], \quad (49)$$

cannot be described by any local realistic model. This naturally implies that such states are also not separable.

In this general scenario, where arbitrary Hardy states belonging to $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ are mixed with a completely chaotic noise, a (necessary) condition, like that of Ref. [17], on the values of p for the existence of a local realistic model, is not known yet. As a consequence, contrary to what happened when dealing with the two-qubit case where the result proven in Ref. [17] could have been applied, it is not possible to compare the strength of our result with some (alternative) criterion based on the violation of a Bell-like inequality, because the latter has not been discovered yet. Thus, we have succeeded to discover whole classes of mixed states, like those of Eq. (33) where $|\phi\rangle$ is any Hardy state whatsoever, such that (i) no local model may exist for them, (ii) they are proven to be nonseparable, without having considered any criterion based on Bell inequalities.

Moreover, in this paper we have considerably enlarged, with respect to the results obtained in Ref. [11], the interval of values for the parameter p such that the corresponding mixed states do not admit any local hidden variable model. In fact, in Ref. [11] a general argument to establish the nonlocal features of certain classes of mixed states, involving set-theoretic techniques similar to those we used in this paper, has been exhibited. In that paper we have proved that, given the Hardy state $|\phi\rangle$ of Eq. (32) and an arbitrary mixed state σ whose trace distance $D(\sigma, |\phi\rangle\langle\phi|) = \frac{1}{2} \text{Tr}[(\sigma - |\phi\rangle\langle\phi|)]$ from the pure state $|\phi\rangle\langle\phi|$ we have denoted as η , if

$$0 \leq \eta < \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{6(p_1^2 + p_2^2 - p_1 p_2)^2}, \quad (50)$$

then no local realistic model exists for σ . The idea of Ref. [11] was that in a definite neighborhood - with respect to the topology induced by the consideration of the trace distance - of a Hardy state there exist uncountable many mixed states which exhibit nonlocal features, and we succeeded in determining the size of that neighborhood. Unfortunately, the result of Eq. (50) was not an optimal one because it did not rely directly on the specific form of the mixed state σ : in fact, in order to be completely general, in Ref. [11] we resorted to majorizations, based on the consideration of the trace distance, for the probability distributions of Eqs. (34)-(39). On the contrary, the method we have presented in this paper makes explicit use of the expression of σ of Eq. (33), which represents a Hardy state corrupted by a white noise, to calculate exactly the relevant probability distributions. As a consequence, it provides us with a larger range of values of the parameter p for which the corresponding statistical operators do not admit a local realistic model, just because the proof relies on the specific properties of σ . In fact, given the state of Eq. (33), we may easily evaluate

$$D(\sigma, |\phi\rangle\langle\phi|) = (1-p) \frac{d_1 d_2 - 1}{d_1 d_2} \equiv \eta \quad (51)$$

and, due to the result of Eq. (50), this implies that whenever $p \in (1 - \frac{d_1 d_2}{6(d_1 d_2 - 1)} \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(p_1^2 + p_2^2 - p_1 p_2)^2}, 1]$ the associated statistical operators cannot be mimicked by local deterministic hidden variable models. The fact that the criterion

presented in this paper is more powerful than the one of Ref. [11] is apparent since the inequality

$$\frac{1}{1 + \frac{d_1 d_2 p_1^2 p_2^2 (p_1 - p_2)^2}{4(p_1^2 + p_2^2 - p_1 p_2)^2}} < 1 - \frac{d_1 d_2}{6(d_1 d_2 - 1)} \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(p_1^2 + p_2^2 - p_1 p_2)^2} \quad (52)$$

holds for single-particle Hilbert spaces of dimension d_i greater or equal to 2 and for any value of $p_1, p_2 \in [0, 1]$ and its validity may be established by very simple analytical calculations. Yet, to better appreciate the improvement with respect to the result of Ref. [11], we have plotted the values of $1 - p = 1 - \frac{1}{1 + \frac{d_1 d_2 p_1^2 p_2^2 (p_1 - p_2)^2}{4(p_1^2 + p_2^2 - p_1 p_2)^2}}$ (the upper curves) and of $1 - p = \frac{d_1 d_2}{6(d_1 d_2 - 1)} \frac{p_1^2 p_2^2 (p_1 - p_2)^2}{(p_1^2 + p_2^2 - p_1 p_2)^2}$ (the lower curves) providing evidence of nonlocality, versus the parameter p_1 for both criteria and for arbitrarily chosen values of $d_1 d_2$ and p_2 :

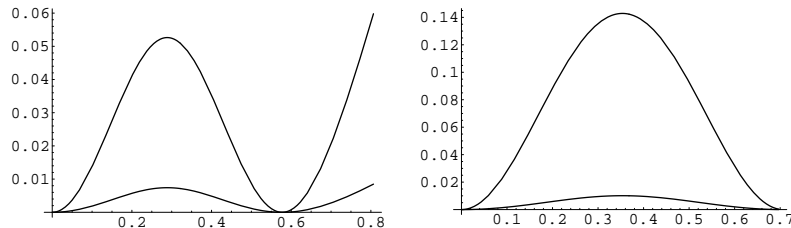


FIG. 1: Values of $1 - p$ versus p_1 , for $d_1 d_2 = 6$ and $p_2 = 1/\sqrt{3}$ (left figure) and for $d_1 d_2 = 12$ and $p_2 = 1/\sqrt{2}$, with the constraint that $p_1^2 + p_2^2 \leq 1$.

Since the greater are the values of $1 - p$ (plotted in the vertical axis) the larger is the set of mixed states exhibiting nonlocal features, the class of mixed states not admitting local descriptions discovered in this paper (the upper curves in Fig.1) is appreciably bigger than the one obtained through the trace-distance method of Ref. [11] (the lower curves in Fig.1). Similar results can be obtained for any choice of $d_1 d_2 > 2$ and for any p_2 . The considerable improvement we have achieved here had to be expected because in Ref. [11] we have not restricted in any way the nature of the noise.

IV. CONCLUSIONS

We have shown how a refinement and a generalization of the original Hardy proof of nonlocality without inequalities can be used to deny the existence of local and deterministic hidden variable models for some one-parameter classes of mixed statistical operators. Such classes contain convex mixtures of pure Hardy states with a completely chaotic noise or a colored noise, and they represent typical mixed states which are considered in the experimental realizations of nonlocality tests. We have explicitly exhibited precise ranges of values of the parameter measuring the amount of noise affecting a nonoptimal preparation of a (pure) Hardy state for which nonlocal effects can be still experimentally revealed and, in some cases, we have bettered the results obtained in Ref. [11].

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